

## SYMMETRIES OF SURFACES

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## §1. INTRODUCTION

1.1. LET  $\sum_g$  denote a compact orientable surface of genus  $g$ . Without explicit mention we shall only consider the effective and orientation-preserving actions of finite groups  $G$  on  $\sum_g$ . For brevity we shall simply say that  $G$  acts on  $\sum_g$  or that  $G$  is a symmetry group of  $\sum_g$ . The cases  $g = 0$  and  $1$  are well-known. For the latter case cf. for example [3], chapter 19. So henceforth we shall also assume  $g \geq 2$ . In this case it is well-known that each  $\sum_g$  admits only finitely many symmetry groups. Some experimentation shows however that their enumeration runs into difficult number-theoretic and finite-group-theoretic problems. In this note we investigate the problem in the reverse direction, namely we fix a finite group  $G$  and attempt an enumeration of those  $g$ s for which  $G$  is a symmetry group of  $\sum_g$ . It turns out that the 2-Sylow subgroup  $G_2$  plays a special role. According to a certain structural property of  $G_2$ , cf. (2.2),  $G$ 's are divided into two types I and II. If  $p^n$  is the order of a  $p$ -Sylow subgroup  $G_p$  of  $G$  and  $p^{e_p}$  is the exponent of  $G_p$  then the integer  $n_p - e_p$  is called the *cyclic  $p$ -deficiency* of  $G$ .

1.2. MAIN THEOREM. *Let  $G$  be a finite group. There exists an integer  $N$  depending only on the cyclic  $p$ -deficiencies of  $G$  with the following property.*

1.2.1. If  $G$  acts on  $\sum_g$  then  $g \equiv 1 \pmod{N}$ , (resp.  $2N$ ), according as  $G$  is of type I (resp. II).

1.2.2. Conversely, if  $G$  is of type I, (resp. II), and  $g \equiv 1 \pmod{N}$ , (resp.  $2N$ ), then  $G$  acts on  $\sum_g$  except for finitely many exceptional values of  $g$ .

For the definition of  $N$  see (2.1). The exceptions mentioned in (2) depend on the solvability in non-negative integers of a certain linear diophantine equation. This equation depends in an essential way on the structure of the group.

1.3. The method of proof of (1.2) combined with some results from the finite group theory provides some precise information on surface-symmetries. The sophisticated results from the finite group theory which we mention follow easily from the classification of finite simple groups. But partly they were in fact some of the initial steps in this classification program.

1.3.1.  $G$  is a symmetry group of almost all (i.e. all but finitely many)  $\sum_g$  iff  $G_p$  is cyclic for  $p$  odd and  $G_2$  is cyclic, dihedral, generalized quaternion or semi-dihedral. Curiously, the family of groups mentioned in this statement includes the finite subgroups of  $SO(3)$  and  $SU(2)$ , the finite groups acting freely on complexes homotopic to spheres, the semi-direct products of two finite cyclic groups of co-prime orders, etc. The only perfect groups among these are  $SL_2(\mathbb{F}_q)$  and  $PSL_2(\mathbb{F}_q)$ ,  $q = a$  prime  $\geq 5$ .

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1.3.2. The only simple symmetry groups of  $\sum_g$ ,  $g$  even, come from:

- (a) three infinite families: (i)  $PSL_2(\mathbb{F}_q)$ ,  $q$  odd  $\geq 5$ , (ii)  $PSL_3(\mathbb{F}_q)$ ,  $q \equiv -1(4)$ , and (iii)  $PSU_3(\mathbb{F}_q)$ ,  $q \equiv 1(4)$ , and
- (b) two special groups: (i)  $A_7$ , and (ii)  $M_{11}$ .

To get an overall understanding of the simple symmetry groups of  $\sum_g$  it is useful to consider the subsets  $S(r) = \{g | g \equiv 1 \pmod{2^r}\}$ . We have  $S(1) \subseteq S(2) \subseteq \dots$  and the result mentioned above refers to  $g \in S(1)$ . For each fixed  $r$  the simple symmetry groups of  $\sum_g$ ,  $g \in S(r)$ , again come from (i) finitely many infinite families of Lie type where the congruences on  $q$  are dictated by the fact that the cyclic 2-deficiency is bounded by  $r$ , and (ii) finitely many alternating, sporadic and other classical groups.

1.4. Besides the structural properties of  $p$ -groups acting on  $\sum_g$  which follow from (1.2) we also derive bounds on their orders by a Hurwitz's-type analysis, cf. (3.5). The bounds for  $p$ -groups are substantially better than Hurwitz's "84 ( $g-1$ )" which applies to all symmetry groups of  $\sum_g$ . This considerably narrows down a list of possible  $p$ -groups acting on  $\sum_g$ , and in certain cases actually leads to their classification. (For example the  $p$ -groups acting on  $\sum_2$  are  $\mathbb{Z}_3$ ,  $\mathbb{Z}_5$  and subgroups of  $SD_{16}$ ; those on  $\sum_4$  are  $\mathbb{Z}_3$ ,  $\mathbb{Z}_5$ ,  $\mathbb{Z}_9$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_3$  and subgroups of  $SD_{32}$ .) It seems possible to carry out the classification of simple groups or  $p$ -groups acting on  $\sum_g$  for small values of  $g$ .

1.5. There are some purely finite-groups-theoretic consequences of this method. So far, they have been rather elementary but their "geometric" proofs are intriguing. For example

1.5.1. A finite perfect group is of type I. This result appears to be new. D. Gorenstein pointed out to me a purely group-theoretic proof of (1.5.1) using transfer. It appears that certain arguments in finite group theory can be interpreted geometrically in terms of realizability of a branchdata for a finite-group-action on a surface.

1.6. *A historical note.* The first results on surface-symmetries ( $g \geq 2$ ) are due to Schwarz, Klein, von Dyck and Hurwitz. Perhaps the most celebrated among these is the Hurwitz's "84( $g-1$ ) theorem" cf. [6]. Partially more precisely Wiman [19, 20] showed that a cyclic symmetry group of  $\sum_g$  has order at most  $4g+2$  and the bound is attained for each  $g$ . Accola [1] and Maclachlan [13] independently showed that  $8(g+1)$  is the minimum of the  $\max |G|$  where  $G$  is a symmetry group of  $\sum_g$ , and for infinitely many  $g$  this is actually the  $\max |G|$  also. The Hurwitz's bounds are attained for infinitely many  $g$ , but the genera for which they are attained are very thinly distributed, cf. [11], [8]. There is a vast literature on surface-symmetries expressed in the language of fuchsian groups, cf. e.g. [4], [8], [11], [12], [17] and the references there. The ubiquitous  $PSL_2$ s abundantly make their appearance in this literature. Nielsen, cf. [15] obtained a topological classification of cyclic symmetry groups. This work is partially extended in [5], [18]. The algebraic geometers have studied surface-symmetries in the context of automorphism groups of function-fields in one variable cf. e.g. [9], [10], [19], [20]. Surface-symmetries also account for the singularities of the moduli spaces of Riemann surfaces, [7], [16].

## §2. CONGRUENCES SATISFIED BY THE GENERA

2.1. Let  $G$  be a finite group of order  $d$ . Let  $\delta$  (resp.  $\Delta$ ) denote the set of all primes dividing  $d$  (resp. prime-powers  $q$  such that  $G$  contains an element of order  $q$ ). Let  $G_p$  denote a Sylow  $p$ -subgroup of  $G$ . The maximum of the orders of the elements of  $G_p$  is called the  $p$ -exponent of  $G$ .

Throughout we write  $|G_p| = p^{n_p}$  and the  $p$ -exponent  $= p^{e_p}$ . The number  $n_p - e_p$  is called the *cyclic  $p$ -deficiency* of  $G$ . Set

$$2.1.1. \quad f_p = \begin{cases} n_p - e_p, & \text{if } p \text{ is odd, or } p = 2 \text{ and } n_2 = e_2 \\ n_2 - e_2 - 1, & \text{if } p = 2 \text{ and } n_2 > e_2 \end{cases}$$

and

$$2.1.2. \quad N = \prod_{p \in \delta} p^{f_p}.$$

2.2. A finite 2-group  $G$  is said to be of *type I* if it is either cyclic or else the elements of order  $< 2^{e_2}$  do not form a subgroup of index 2. Otherwise  $G$  is said to be of *type II*. In other words  $G$  is of type II if it is not cyclic and there exists a homomorphism  $\varphi: G \rightarrow \mathbb{Z}_2 = \{0, 1\}$  such that  $\varphi^{-1}(1)$  consists of elements of order  $2^{e_2}$  and  $\varphi^{-1}(0)$  consists of elements of order  $< 2^{e_2}$ . More generally a finite group  $G$  will be said to be of type I (resp. II) if  $G_2$  is of type I (resp. II).

2.3. THEOREM. *Let  $G$  be a finite group.*

*Using the notation and terminology introduced above we let  $S$  denote the set of all integers  $\geq 2$  and  $\equiv 1(N)$  (resp.  $(2N)$ ) if  $G$  is of type I (resp. II). Then*

- (a)  $G$  acts on  $\sum_g$ ,  $g \geq 2 \Rightarrow g \in S$
- (b) For all but finitely many  $g \in S$ ,  $G$  acts on  $\sum_g$ .

2.4. The proof of (2.3) is divided into several parts. First note that if  $G$  acts on  $\sum_g$  then  $G \setminus \sum_g \approx \sum_h$  for some  $h$ . Let  $p: \sum_g \rightarrow \sum_h$  denote the canonical projection and call

$$2.4.1. \quad S = \{x \in \sum_g \mid G_x \neq \{1\}\}$$

the *singular set* of the  $G$ -action. The set  $B = p(S)$  is called the *branch set* of the  $G$ -action. For  $x \in B$  and any  $y \in p^{-1}(x)$  the integer  $n_x = |G_y|$  is called the *branching-index* at  $x$ . The well-known Riemann–Hurwitz relation reads

$$2.4.2. \quad 2 - 2g = d \left\{ 2 - 2h - \sum_{x \in B} \left( 1 - \frac{1}{n_x} \right) \right\}.$$

2.5. First consider the case when  $G$  is a  $p$ -group. Write  $n, e, f$  for  $n_p, e_p, f_p$ , respectively. If  $G$  acts on  $\sum_g$  and there are  $a_j$  branch points of branching index  $p^j$  then (2.4.2) takes the form

$$2.5.1. \quad 2 - 2g = p^n(2 - 2h) - \sum_{j=1}^e (p^j - 1)p^{n-j}a_j.$$

This shows that if  $p$  is odd then  $p^{n-e}$  divides  $g - 1$  and if  $p = 2$  and  $n > e$  then  $2^{n-e-1}$  divides  $g - 1$ . By the definitions of  $f$  and  $N$  it follows that  $g \equiv 1(N)$ . In the general case when  $G$  is not necessarily a  $p$ -group the same result follows from the existence of Sylow  $p$ -subgroups and the Chinese remainder theorem.

2.6. Now suppose that  $G$  is a 2-group of type II. We again write  $n, e, f$  for  $n_2, e_2, f_2$ , respectively. By (2.5) we know that  $g \equiv 1(2^f)$ . Our claim now is that in fact  $g \equiv 1(2^{1+f})$ . Suppose this is not the case, i.e.  $g \not\equiv 1(2^{1+f})$ . Then (2.5.1) shows that  $a_e$  is odd. By the well-known description of finite-group actions on surfaces going back to Hurwitz cf. [3] p. 420, it follows that a necessary and sufficient condition for the existence of such a  $G$ -action is the

existence of generators  $u_i, v_i, 1 \leq i \leq h$  and  $x_{jk}, 1 \leq j \leq e, 1 \leq k \leq a_j$  of  $G$  such that  $x_{jk}$  has order  $2^j$  and

$$2.6.2. \quad \prod_{i=1}^h (u_i, v_i) \prod_{j=1}^e \prod_{k=1}^{a_j} x_{jk} = 1.$$

By hypothesis there exists a homomorphism  $\varphi: G \rightarrow \mathbb{Z}_2 = \{0, 1\}$  such that  $\varphi^{-1}(1)$  consists of elements of order  $2^e$  whereas  $\varphi^{-1}(0)$  consists of elements of order  $< 2^e$ . It is also clear that  $\varphi^{-1}(0)$  contains all commutators. But now applying  $\varphi$  to both sides of (2.6.2) and noting again that  $a_e$  is odd, we clearly get a desired contradiction. This proves the part (a) of (2.3).

2.7. Now we proceed to prove the part (b) of (2.3). For  $n \in \Delta$  set  $n' = d/n$ . The formula (2.4.2) suggests considering the diophantine equation

$$2.7.1. \quad \sum_{n \in \Delta} (n-1)n'x_n + 2dy = 2(z-1) + 2d$$

to be solved for  $x_n, y$  and  $z$  in non-negative integers. The *g.c.d.* of the coefficients of the L.H.S. of (2.7.1) is  $N$  if  $n_2 = e_2$  and  $2N$  if  $n_2 > e_2$ . So for a solution of (2.7.1) we must have  $z \equiv 1(N)$ . It is an elementary fact from number theory that if  $g \equiv 1(N)$  and is sufficiently large then (2.7.1) with  $z = g$  can be solved for  $x_n$  and  $y$  in non-negative integers, or more generally  $x_n$  and  $y$  can be chosen to be greater than some fixed values. It would suffice to show that if  $z = g, x_n = a_n, y = h$  is such a solution of (2.7.1), and  $g \in S$ , then we can choose the elements  $u_i, v_i, x_{n_1}, x_{n_2}, \dots, x_{na_n}, 1 \leq i \leq h, n \in \Delta$  in  $G$  so that (2.6.2) holds. The lower bound for the values of  $g$  and  $a_n$  needed for this purpose will be indicated in the course of the proof.

2.8. First set  $u_i = v_i = 1$ . Next note that  $G$  certainly admits a set of generators whose orders are prime-powers. If  $\{e_1, \dots, e_s\}$  is such a system we make sure that (2.6.2) contains a segment of the form  $e_1 e_1^{-1} e_2 e_2^{-1} \dots e_s e_s^{-1}$ .<sup>\*</sup> In this process for each  $n \in \Delta$  we have used 0 or 2 slots out of the  $a_n$  allowable slots.

2.9. Now suppose  $n$  is a power of an odd prime  $p$ . If  $a_n$  is even then we again insert segments of the form  $u \cdot u^{-1}$  in (2.6.2). If  $a_n$  is odd, then we assume that  $a_n \geq 5$ . So after possibly using 2 slots for the procedure described in (2.8) at least 3 slots remain. Now note that an element of order  $p^r, p = \text{an odd prime}$ , can always be written as a product of two elements of order  $p^r$ . So the remaining slots can be filled in by a segment of elements of order  $n$  whose product is identity.

2.10. Now let  $n$  be a power of 2. Let  $G_2$  be a Sylow 2-subgroup and write  $r = |G_2|$  and  $s = \text{the 2-exponent of } G$ . We need to consider 3 cases.

*CASE (1).  $r = s$ : Then  $G_2$  is cyclic. The coefficient of  $x_r$  in (2.7.1) is odd, and it is the only odd coefficient. So  $a_r$  must be even. In  $G_2$  any element of order  $< r$  may be written as product of two generators of  $G_2$ . So if some  $a_t, t = 2^j < r$  is odd we use two slots in  $a_r$  and one in  $a_t$  to insert in (2.6.2) a segment of three elements (one of order  $t$  and two of order  $r$ ) whose product is identity. The remaining  $a_t - 1$  slots may be filled by  $u \cdot u^{-1}$ -type segments. This process repeated for each  $t = 2^j < r$  takes up only even number of slots in  $a_r$ . (So we need to ensure that  $a_r$  is sufficiently large for this purpose.) The remaining of the  $a_r$  slots may be filled by  $u \cdot u^{-1}$ -type segments.*

<sup>\*</sup>Note that the order of the factors in (2.6.2) is immaterial.

CASE (2).  $r > s$ ,  $G$  of type 2: Then by choice  $g \equiv 1(2N)$  and so  $a_r$  is even hence the process described in case 1 may be repeated.

CASE (3).  $r > s$ ,  $G$  of type 1: If  $a_r$  is even, again we can proceed as above. So suppose  $a_r$  is odd. This occurs precisely when  $g \equiv 1(N)$  but  $g \not\equiv 1(2N)$ . Let  $P$  be the subgroup of  $G_2$  generated by elements of order  $< r$ . Clearly  $P$  is normal and  $G_2/P$  is an elementary abelian 2-group. If  $P = G_2$  then an element of order  $r$  can be expressed as a product of elements of order  $< r$ . So we can use one slot in  $a_r$  and certain others in  $a_t$ ,  $t = 2^j < r$  to be filled by a segment of elements (one of order  $r$  and the others of order  $< r$ ) whose product is identity. If  $P \neq G_2$  then it has index at least 4. So it is possible to express an element of order  $r$  as a product of two elements of order  $r$ . Thus we can use 3 slots in  $a_r$  to be filled by elements of order  $r$  whose product is identity. In any case there are even number of slots remaining in  $a_r$ . So we can proceed further if necessary as in case (1).

This finishes the proof of the Theorem (2.3).

2.11. Remark. For a precise enumeration of  $g$  such that  $G$  acts on  $\sum_g$  we shall need to solve the equation analogous to (2.7.1) where  $\Delta$  is replaced by the set of all orders of elements of  $G$ .

### §3. APPLICATIONS

3.1. PROPOSITION. Let  $G$  be a symmetry  $p$ -group of  $\sum_g$ , and  $p \nmid (g-1)$ .

- (a) If  $p$  is odd then  $G$  is cyclic.
- (b) If  $p = 2$  then  $G$  is cyclic, dihedral, generalized quaternion or semi-dihedral.

(Recall that by definition, a dihedral group

$$D_{2^{n+1}} = \langle x, y | x^{2^n} = e = y^2, yxy^{-1} = x^{-1} \rangle, \quad n \geq 1,$$

a generalized quaternion group

$$Q_{2^{n+1}} = \langle x, y | x^{2^n} = e, x^{2^{n-1}} = y^2, yxy^{-1} = x^{-1} \rangle, \quad n \geq 2,$$

a semi-dihedral group

$$SD_{2^{n+1}} = \langle x, y | x^{2^n} = e = y^2, yxy^{-1} = x^{2^{n-1}-1} \rangle, \quad n \geq 3,$$

and a twisted dihedral group

$$S_*D_{2^{n+1}} = \langle x, y | x^{2^n} = e = y^2, yxy^{-1} = x^{2^{n-1}+1} \rangle, \quad n \geq 3.)$$

*Proof.* The hypotheses imply that in the notation of (2.1)  $f_p = 0$ . So  $G$  has cyclic deficiency 0 if  $p$  is odd and  $\leq 1$  if  $p = 2$ . So  $G_p$  is cyclic for  $p$  odd. The 2-groups with cyclic deficiency  $\leq 1$  are well-known. Besides those stated in the proposition they are  $\mathbb{Z}_{2^n} \oplus \mathbb{Z}_2$ ,  $n \geq 2$  and  $S_*D_{2^{n+1}}$ ,  $n \geq 3$ . A calculation for both of these types shows that they are of type II, and so by (2.3) they cannot act on  $\sum_g$ , if  $g$  is even.

q.e.d.

3.2. PROPOSITION. Let  $G$  be a  $p$ -group acting on  $\sum_g$  and  $p^r$  the highest power of  $p$  dividing  $g-1$ . Then the cyclic deficiency of  $G$  is  $\leq r$  if  $p$  is odd, and  $\leq r+1$  if  $p = 2$ . In particular if  $G$  is an elementary abelian  $p$ -group of rank  $s$  then  $s \leq r+1$  if  $p$  is odd, and  $\leq r+2$  if  $p = 2$ .

*Proof.* The first assertion follows immediately from (2.3). If  $G$  is an elementary abelian  $p$ -group of rank  $s$  then its cyclic deficiency is  $s - 1$ . Hence the second assertion.

q.e.d.

3.3. PROPOSITION. Let  $G$  be a cyclic  $p$ -group acting on  $\sum_g$  then  $|G| \leq \frac{2p}{p-1}g$ .

*Proof.* Let  $G \backslash \sum_g \approx \sum_h$  with  $a_j$  branch points of branching index  $p^j$ . Then by the Riemann–Hurwitz formula

$$2 - g = |G| \left\{ (2 - 2h) - \sum_j a_j \left( 1 - \frac{1}{p^j} \right) \right\}.$$

If  $h \geq 2$  then  $2 - 2g \leq |G|(2 - 2h) \leq -2|G|$  or  $|G| \leq g - 1$ . If  $h = 1$  there are at least two branch points and we have  $|G| \leq \frac{p}{p-1}(g - 1)$ . Now suppose  $h = 0$ . Then there are at least 3 branch points and at least two branching indices equal  $|G|$ . So

$$2 - 2g \leq |G| \left\{ 2 - 2 \left( 1 - \frac{1}{|G|} \right) - \left( 1 - \frac{1}{p} \right) \right\} = 2 - |G| \frac{p-1}{p}$$

or 
$$|G| \leq \frac{2p}{p-1}g.$$

q.e.d.

3.4. PROPOSITION. Let  $G$  be a non-cyclic  $p$ -group acting on  $\sum_g$ .

Let  $k \geq 2$  be the minimum number of generators of  $G$ . Then

$$|G| \leq \begin{cases} 16(g-1) & \text{if } k = 2, p = 2, \\ 9(g-1) & \text{if } k = 2, p = 3, \\ 8(g-1) & \text{if } k = 3, p = 2, \\ \frac{2p(g-1)}{(k-1)p - (k+1)} & \text{otherwise.} \end{cases}$$

*Proof.* Let  $G \backslash \sum_g \approx \sum_h$ . If  $k \leq 2h - 1$  then  $h \geq 2$  and by the Riemann–Hurwitz formula we get

$$2 - 2g \leq |G|\{2 - 2h\} \leq |G|\{1 - k\}, \quad \text{i.e.} \quad |G| \leq \frac{2}{k-1}(g-1).$$

So suppose  $k \geq 2h$ . We now use the fact that a minimal set of generators of a  $p$ -group maps injectively onto a minimal set of generators of its commutator quotient. So we see that there must be at least  $k - 2h + 1$  branch points. So

$$2 - 2g \leq |G| \left\{ 2 - 2h - (k - 2h + 1) \left( 1 - \frac{1}{p} \right) \right\}, \quad \text{or}$$

$$3.4.1. \quad |G| \leq \frac{2p(g-1)}{(k-1)p - (k-2h+1)} \leq \frac{2p(g-1)}{(k-1)p - (k+1)}$$

where the inequalities are valid as long as the denominators are positive. The denominator of the last term is already positive for all pairs  $(p, k)$  except  $(p, k) = (3, 2), (2, 3)$  and  $(2, 2)$ . We analyze these exceptional cases separately.

CASE (i),  $p = 3, k = 2$ . If  $h \geq 1$  the denominators of the middle term of (3.4.1) are positive. This gives an estimate  $|G| \leq 3(g-1)$ . Now suppose  $h = 0$ . Then there are at least 3 branch points. If there are  $\geq 4$  branch points we get an estimate

$$2 - 2g \leq |G| \{2 - 4(1 - \frac{1}{3})\} \quad \text{or} \quad |G| \leq 3(g-1).$$

If there are 3 branch points then the branching indices  $\{3, 3, 9\}$  give the best upper estimate

$$2 - 2g \leq |G| \{2 - 2(1 - \frac{1}{3}) - (1 - \frac{1}{9})\}, \quad \text{or} \quad |G| \leq 9(g-1).$$

CASE (ii),  $p = 2, k = 3$ . If  $h \geq 1$  the middle estimate in (3.4.1) gives  $|G| \leq 2(g-1)$ . If  $h = 0$  then there are at least 4 branch points, and the branching indices  $\{2, 2, 2, 4\}$  give the best upper estimate  $|G| \leq 8(g-1)$ .

CASE (iii),  $p = 2, k = 2$ . The analysis as in the above cases shows that the branching indices  $\{2, 4, 8\}$  give the best upper estimate  $|G| \leq 16(g-1)$ .

q.e.d.

3.5. COROLLARY. If  $G$  is a  $p$ -group acting on  $\Sigma_g$  then

$$|G| \leq \begin{cases} 16(g-1) & \text{if } p = 2, \\ 9(g-1) & \text{if } p = 3, \\ \frac{2p(g-1)}{p-3} & \text{if } p \geq 5. \end{cases}$$

Moreover for each prime  $p$  these bounds are attained for infinitely many values of  $g$ , and also not attained for infinitely many values of  $g$ .

*Proof.* Comparing (3.3) and (3.4) we see that for  $p = 2$  and 3 bounds for the 2-generator groups are higher than those for  $k$ -generator groups,  $k \neq 2$ . For  $p \geq 5$  one sees that the bound  $\frac{2p}{p-3}(g-1)$  for a 2-generator group is higher than that for a  $k$ -generator group,  $k \geq 3$ , and moreover  $\frac{2p}{p-1}g \leq \frac{2p}{p-3}(g-1)$  as long as  $p \leq 2g+1$ . On the other hand (3.3) applied to  $G = \mathbb{Z}_p$  does imply  $p \leq 2g+1$ . So  $\frac{2p}{p-3}(g-1)$  is the estimate for  $|G|$  for  $p \geq 5$ . The proof of (3.3), (3.4) also shows that the bounds are attained precisely when  $G \backslash \Sigma_g \approx \Sigma_0$ , with three branch points with branching indices  $(2, 4, 8)$  for  $p = 2$ ,  $(3, 3, 9)$  for  $p = 3$ , and  $(p, p, p)$  for  $p \geq 5$ . Interpreted in terms of fuchsian groups this means that the bounds are attained for  $G$ s which are quotients of the Schwarz triangle groups  $\Delta_{2,4,8}$  for  $p = 2$ ,  $\Delta_{3,3,9}$  for  $p = 3$  and  $\Delta_{p,p,p}$  for  $p \geq 5$  with torsion-free kernel. One easily sees that there are infinitely many such possibilities. Also there are infinitely many genera not admitting such actions since existence of such actions implies non-trivial congruences on  $g$ . (In fact the genera admitting such actions have natural density zero in the set of positive integers.)

q.e.d.

3.6. *Remark.* The  $p$ -groups with cyclic deficiency  $\leq 2$  are known, cf. [3], [14]. So if  $g-1$  is not divisible by 4 or by  $p^3$ ,  $p$  odd, then using (3.1)–(3.5) one has a small list of  $p$ -groups using which one can actually enumerate the  $p$ -groups acting on  $\Sigma_g$ . Similar remarks apply to the actions of finite simple groups on  $\Sigma_g$ , cf. (1.3), and (3.7) below. (The topological equivalence

classes of the actions can be determined by Hurwitz's procedure, cf. (2.6). Each such topological equivalence class defines a certain subvariety of the Teichmüller space of Riemann surfaces of genus  $g$ . The topology of this subvariety can be determined, in principle, from the branching data of the action.)

3.7. *Remark.* The proof of the assertion in (1.3.2) in the introduction follows from (3.1) and [2] which contains a classification of finite simple groups whose 2-Sylow subgroups contain an elementary abelian 2-group of rank at most 2. For the latter remarks we have to appeal to the full classification of finite simple groups. For instance if  $g \neq 1$  (4) then the only simple symmetry groups of  $\sum_g$ , besides the ones listed (1.3.2), are the first Janko group  $J_1$ , the Ree groups  ${}^2G_2(q)$ ,  $q = 3^{2n+1}$ ,  $n \geq 2$ , and  $PSL_2(\mathbb{F}_8)$ . These results may be further extended in the following direction. Let  $G$  be a finite perfect group acting on  $\sum_g$ ,  $g$  even, and  $O(G)$  = the maximal normal subgroup of odd order in  $G$ . Then  $G/O(G)$  is isomorphic to one of the groups listed at the beginning of (1.3.2) or else it is isomorphic to  $SL_2(\mathbb{F}_q)$   $q$  odd  $\geq 5$  or  $A^* =$  the 2-fold central prefect extension of  $A^7$ .

Notice that if  $G \setminus \sum_g$  has genus 0 and the branching indices are pairwise co-prime then  $G$  is necessarily perfect. The above remarks indicate the very special nature of such actions.

3.8. *Remark.* The proof of (1.3.1) in the introduction follows from (3.1), (3.7) and a simple check that in the list given in (1.3.2) only  $PSL_2(\mathbb{F}_q)$   $q = a$  prime  $\geq 5$  has cyclic  $p$ -Sylow subgroups for  $p$  odd, and  $SL_2(\mathbb{F}_q)$   $q = a$  prime  $\geq 5$  is its only perfect extension with the same property.

Now we mention two purely group-theoretic consequences, which are presented mainly for the novelty of their "geometric" proofs.

3.9. PROPOSITION. *Let  $G$  be a  $p$ -group and  $H \leq G$ .*

*Then the cyclic deficiency of  $H$  is less than or equal to that of  $G$ . In particular the rank of an elementary abelian subgroup is  $\leq 1 + \{\text{the deficiency of } G\}$ .*

*Proof.* Suppose  $p$  is odd. By (2.3) we may choose  $g$  such that  $G$  acts on  $\sum_g$  and the highest power of  $p$  dividing  $g - 1$  equals the cyclic deficiency of  $G$ . But then  $H$  acts on  $\sum_g$  and the result is clear from (2.3). A slight modification of this argument for  $p = 2$  is left to the reader. The last assertion is clear.

q.e.d.

3.10. PROPOSITION. *A finite perfect group is of type I.*

*Proof.* Let  $G$  be a finite perfect group of order  $d$  and suppose if possible that it is of type II. We use the notations of §2. For  $z = g \equiv 1(N)$  but  $\not\equiv 1(2N)$ , and  $g$  sufficiently large we can find a solution of (2.7.1) in non-negative integers for  $x_n$ ,  $y$  and  $h$ , and we may assume that  $h$  is sufficiently large. According to (2.3)  $G$  cannot act on  $\sum_g$ . On the other hand in a perfect group the equation (2.6.2) clearly has a solution for large  $h$  so if  $h$  is sufficiently large then  $G$  would act on  $\sum_g$ . This contradiction shows that  $G$  must be of type I.

q.e.d.

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## REFERENCES

1. R. D. M. ACCOLA: On the number of automorphisms of a closed Riemann surface, *Trans. A.M.S.* **131** (1968), 398–408.
2. A. J. ALPERIN, R. BRAUER and D. GORENSTEIN: Finite simple groups of 2-rank two, *Scripta Math.* **29** (1973), 191–214.
3. W. BURNSIDE: Theory of groups of finite order. Dover (1955).
4. J. E. COHEN: Homomorphisms of co-compact fuchsian groups on  $\mathrm{PSL}_2(\mathbb{Z}_p[x]/f(x))$ , *Trans. A.M.S.* **281** (1984), 571–585.
5. A. EDMONDS: Surface-symmetry I and II, *Mich. Math. J.* **29** (1982), 171–183, and **30** (1983), 143–154.
6. A. HURWITZ: Über algebraische Gebilde mit eindeutigen Transformationen in sich, *Math. Ann.* **41** (1893), 403–442.
7. J.-I. IGUSA: Arithmetic variety of moduli for genus two, *Ann. of Math.* **72** (1960), 612–649.
8. R. S. KULKARNI: Normal subgroups of fuchsian groups, *Q. J. Math. Oxford* **36** (1985), 325–344.
9. A. KURIBAYASHI: On analytic families of compact Riemann surfaces with non-trivial automorphisms, *Nagoya Math. J.* **28** (1966), 119–165.
10. A. KURIBAYASHI: On analytic families of compact Riemann surfaces of genus 3 and the generalized Teichmüller spaces, *J. Math. Soc. Japan* **28** (1976), 712–736.
11. A. M. MACBEATH: On a theorem of Hurwitz, *Proc. Glasgow Math. Asso.* **5** (1961), 90–96.
12. A. M. MACBEATH: Generators of the linear fractional groups, *Proc. Symp. Pure Math.* Vol. XII, Am. Math. Soc. (1969), 14–32.
13. C. MACLACHLAN: A bound for the number of automorphisms of a compact Riemann surface, *J. London Math. Soc.* **44** (1969), 265–272.
14. G. A. MILLER: Determination of all the groups of order  $p^m$  . . . . . *Trans. A.M.S.* **2** (1901), 259–272 and *ibid.* **3** (1902), 383–387.
15. J. NIELSEN: Die Struktur periodischer Transformationen von Flächen, *Danske Vid. Selsk. Mat-Phys. Medd.* **15** (1937), 1–77.
16. H. E. RAUCH: A transcendental view of the space of algebraic Riemann surfaces, *Bull. A.M.S.* **71** (1956), 1–39.
17. C.-H. SAH: Groups related to compact Riemann surfaces, *Acta. Math.* **123** (1969), 13–42.
18. P. A. SMITH: Abelian actions on 2-manifolds, *Mich. Math. J.* **14** (1967), 257–275.
19. A. WIMAN: Über die hyperelliptischen Kurven. . . . . ,” *Bihang Till Kongl. Svenska Vetenskaps Akademiens Handlingar Stockholm* (1895–1896), Vol. 21, No. 1, 1–23.
20. A. WIMAN: “Über die algebraischen Kurven . . . . . ,” *ibid.* **3**, 1–41.

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